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Asymptotic form of the two-point correlation function of the *XXZ* spin chain

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Abstract. Correlation functions of the *XXZ* spin chain in the critical regime are studied at zero temperature. They are exactly represented in the Fredholm determinant form and are related to an operator-valued Riemann–Hilbert problem. Analysing this problem we prove that a two-point correlation function is asymptotically expressed in terms of power functions of the distance between spin operators.

1. Introduction

We consider a two-point correlation function of the *XXZ* spin chain in an external magnetic field at zero temperature. The Hamiltonian is defined by

$$H_{XXZ} = \sum_{n\in\mathbb{Z}}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z - h\sigma_n^z). \tag{1.1}$$

The anisotropy parameter Δ is parametrized as $\Delta = \cos 2\eta$ ($\frac{\pi}{2} < \eta < \pi$), which implies the critical regime. The Pauli matrices σ_n^x, σ_n^y and σ_n^z act on the nth site and h indicates an external magnetic field. The *XXZ* spin chain is one of the quantum solvable models and is analysed by means of the quantum inverse scattering method [1]. Based on this method the generating functional of correlation functions is represented in the Fredholm determinant form [2].

The determinant representation of the generating functional has been investigated in some limiting cases. In the limit of strong magnetic field $h \to h_c = 4\cos^2\eta$ this is connected with the τ function of the Painlevé V equation [3]. On the basis of this equation we can evaluate correlation functions of the *XXZ* spin chain in strong magnetic field [4]. The generating functional also depends on an extra parameter α (see (2.2)) and in the limit $\alpha \to -\infty$ itself indicates a special correlation function called the ferromagnetic-string-formation probability (FSFP). The FSFP describes the probability of finding a ferromagnetic string of adjacent parallel spins and the corresponding Fredholm determinant is known to be related to an operator-valued Riemann–Hilbert problem [5]. Thus the evaluation of FSFP is reduced to the solution of this problem. We remark that either the analysis of the Painlevé V equation or the Riemann–Hilbert problem is a kind of classical inverse scattering problem. For special cases above, the calculation of correlation functions of the *XXZ* spin chain can be reduced to a *classical* inverse problem, although the model is quantum mechanical.

Recently, an operator-valued Riemann–Hilbert problem associated with the generating functional has been proposed [6]. Surprisingly, the calculation of *any* correlation function of the *XXZ* spin chain is interpreted as a classical inverse scattering problem. In particular we are in a position to obtain the closed forms of two-point correlation functions: there was no valid method to evaluate two-point correlation functions of the *XXZ* spin chain in the framework of quantum solvable models, because the integral representations of two-point correlation functions obtained in [7] and [8] are too complicated to analyse them. In this paper, on the basis of our Riemann–Hilbert problem, we compute the large-m asymptotic form of a two-point correlation function $\langle\sigma_m^z\sigma_1^z\rangle$ and prove the following theorem.

Theorem. *For sufficiently large m a two-point correlation function* $\langle\sigma_m^z\sigma_1^z\rangle$ *is expressed in terms of power-functions of m.*

This corresponds to a rigorous proof of the well known conjecture that two-point functions decay as a power of the distance between spin operators at zero temperature in the critical regime, which was derived by means of conformal field theory, the bosonization method, the numerical analysis and so on. In order to prove the theorem, in section 2 we define the Fredholm determinant representation of the generating functional and formulate the associated operator-valued Riemann–Hilbert problem. In section 3 we analyse this problem and determine the asymptotic form of the generating functional. As a result the theorem is proved. Section 4 is devoted to concluding remarks.

2. Operator-valued Riemann–Hilbert problem

In this paper we pay attention to a two-point correlation function $\langle\sigma_m^z\sigma_1^z\rangle$, which is computed through the generating functional. In this section we define the Fredholm determinant representation of the generating functional and formulate the associated operator-valued Riemann–Hilbert problem.

2.1. Fredholm determinant

A two-point correlation function $\langle\sigma_m^z\sigma_1^z\rangle$ is given by the generating functional $Q^{(m)}(\alpha)$ via

$$\langle\sigma_m^z\sigma_1^z\rangle = 2\Delta_m \left.\frac{\partial^2 Q^{(m)}(\alpha)}{\partial\alpha^2}\right|_{\alpha=0} - 4\left.\frac{\partial Q^{(1)}(\alpha)}{\partial\alpha}\right|_{\alpha=0} + 1. \tag{2.1}$$

Here Δ_m is the lattice Laplacian defined by $\Delta_m f(m) = f(m) - 2f(m-1) + f(m-2)$. The generating functional $Q^{(m)}(\alpha)$ is represented by the Fredholm determinant [2]:

$$Q^{(m)}(\alpha) = Q_0\langle\mathrm{vac}|\mathrm{Det}\left(1 - \frac{1}{2\pi}V^{(m)}\right)|\mathrm{vac}\rangle \tag{2.2}$$

where Q_0 is a normalization factor. The Fredholm determinant is expanded as

$$\mathrm{Det}\left(1 - \frac{1}{2\pi}V^{(m)}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{1}{2\pi}\right)^n\int_\Gamma\prod_{k=1}^{n}\mathrm{d}z_k\,\det{}_n V^{(m)} \tag{2.3}$$

where $\det_n V^{(m)}$ is the normal determinant of an $n\times n$ matrix $V^{(m)}(z_i, z_j)$ $(i, j = 1, \ldots, n)$. The integration contour Γ runs on a part of the unit circle anti-clockwise: $\Gamma = \{e^{i\theta}|2\pi - \psi < \theta < \psi\}$. The boundary value ψ depends on the anisotropy parameter η and the external magnetic field h, lies in $\pi < \psi < 2\eta$ and approaches π in the strong-magnetic-field limit $h \to h_c = 4\cos^2\eta$ and 2η in the limit $h \to 0$ [1, 6]. The kernel $V^{(m)}$ is given by the scalar product of four-component vectors as follows [6]:

$$V^{(m)}(z, w) = \mathrm{i}\int_0^\infty \frac{\mathrm{d}s}{z-w}\sum_{j=1}^{4} a_j^{(m)}(z|s)b_j^{(m)}(w|s) \tag{2.4}$$

where vectors are defined ($q = \mathrm{e}^{2\mathrm{i}\eta}$)

$$\boldsymbol{a}^{(m)}(z|s) = \left(\mathrm{i}(q-q^{-1})\frac{qz-1}{z-q}\right)^{1/2} \begin{pmatrix} z^{m/2}\exp(\mathrm{i}q^{-1}\frac{qz-1}{z-q}s+\alpha-\varphi_3(z)) \\ z^{m/2}\exp(-\mathrm{i}q\frac{qz-1}{z-q}s) \\ z^{-m/2}\exp(\mathrm{i}q^{-1}\frac{qz-1}{z-q}s-\varphi_2(z)) \\ z^{-m/2}\exp(-\mathrm{i}q\frac{qz-1}{z-q}s+\alpha+\varphi_1(z)-\varphi_3(z)) \end{pmatrix} \tag{2.5}$$

$$\boldsymbol{b}^{(m)}(z|s) = \left(\mathrm{i}(q-q^{-1})\frac{qz-1}{z-q}\right)^{1/2} \begin{pmatrix} z^{-m/2}\exp(-\mathrm{i}q\frac{qz-1}{z-q}s+\varphi_4(z)) \\ -z^{-m/2}\exp(\mathrm{i}q^{-1}\frac{qz-1}{z-q}s) \\ z^{m/2}\exp(-\mathrm{i}q\frac{qz-1}{z-q}s+\varphi_2(z)) \\ -z^{m/2}\exp(\mathrm{i}q^{-1}\frac{qz-1}{z-q}s-\varphi_1(z)+\varphi_4(z)) \end{pmatrix}. \tag{2.6}$$

The operators $\varphi_j(z)$ ($j = 1, \ldots, 4$) are bosonic quantum fields called the *dual fields*. They are decomposed into coordinate and momentum fields as $\varphi_j(z) = q_j(z) + p_j(z)$ and act on the vacuum states defined by $\langle\mathrm{vac}|q_j(z) = p_j(z)|\mathrm{vac}\rangle = 0$ ($j = 1, \ldots, 4$). The commutation relation between coordinate and momentum fields is expressed in the following matrix form:

$$[q_j(z), p_k(w)] = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}_{jk} \log h(z,w) + \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}_{jk} \log h(w,z) \tag{2.7}$$

with

$$h(z,w) = \frac{1}{q-q^{-1}}\left(q\sqrt{\frac{qz-1}{z-q}\frac{w-q}{qw-1}} - q^{-1}\sqrt{\frac{z-q}{qz-1}\frac{qw-1}{w-q}}\right). \tag{2.8}$$

By definition the dual fields are commutative: $[\varphi_j(z), \varphi_k(w)] = 0$ ($j, k = 1, \ldots, 4$). They are thus interpreted as entire functions unless they act on the vacuum states to produce the vacuum expectation values. The following lemma is useful.

Lemma 1. *A dual field $\varphi_4(z)$ is equivalent to $\varphi_3(z)$.*

Proof. If a set of bosonic fields with four-species has the same commutation relation as (2.7), the kernel $V^{(m)}$ whose dual fields are replaced with those fields is equivalent to the original kernel. We put c-numbers α and $z^{\pm m/2}$ into the dual fields. Under a simultaneous replacement $\varphi_1 \leftrightarrow \varphi_2$ and $\varphi_3 \leftrightarrow \varphi_4$ the commutation relation (2.7) is invariant and the kernel $V^{(m)}(z,w)$ is then changed into $\exp(\varphi_3(w) - \varphi_4(z))V^{(m)}(w,z)$. It thus follows that

$$V^{(m)}(z,z) = \exp(\varphi_3(z) - \varphi_4(z))V^{(m)}(z,z). \tag{2.9}$$

Note that the kernel $V^{(m)}(z,z)$ contains the dual fields $\varphi_1(z), \ldots, \varphi_4(z)$. In order to produce the same vacuum expectation values on both sides, a dual field $\varphi_4(z)$ must be equal to $\varphi_3(z)$. $\square$

Hearafter we replace a dual field $\varphi_4(z)$ with $\varphi_3(z)$ on the basis of this lemma. By virtue of this lemma the recursion relation (3.21) contains no dual fields and makes it clear that two-point correlation functions decay in power.

2.2. *Operator-valued Riemann–Hilbert problem*

The Fredholm determinant $\mathrm{Det}(1 - \frac{1}{2\pi}V^{(m)})$ is related to *matrix integral operators* [6]. They are defined by matrices whose multiplication requires extra integral calculus: those products are formulated as

$$(EF)_{ij}(z|s,t) = \int_0^\infty \mathrm{d}r \sum_k E_{ik}(z|s,r)F_{kj}(z|r,t). \tag{2.10}$$

From now on we omit the dependence on extra variables s, t of matrix integral operators unless necessary and denote the delta function $\delta(s-t)$ by I. Consider a 4×4 matrix integral operator $\chi^{(m)}(z)$ that satisfies the following conditions (a)–(c) [6]:

(a) Each element of $\chi^{(m)}(z|s,t)$ is an analytic function of $z \in \mathbb{C}\backslash\Gamma$ for any $m \in \mathbb{Z}_{>0}$, $s, t \in \mathbb{C}$.
(b) $\chi^{(m)}(\infty) = I$.
(c) Let $\chi_+^{(m)}(z)$ $(\chi_-^{(m)}(z))$ be the boundary value of $\chi^{(m)}(z)$ on the left-hand (the right-hand) side of a positive direction of the oriented contour Γ. It then follows that

$$\chi_-^{(m)}(z) = \chi_+^{(m)}(z)L^{(m)}(z) \qquad (z\in\Gamma). \tag{2.11}$$

The 4×4 matrix integral operator $L^{(m)}(z)$ is called the *conjugation matrix* and is expressed by

$$L^{(m)}(z) = \begin{pmatrix} I-\kappa P(z) & \kappa \mathrm{e}^{-\varphi_3}Q(z) & -\kappa z^m \mathrm{e}^{\varphi_2-\varphi_3}P(z) & \kappa z^m \mathrm{e}^{-\varphi_1}Q(z) \\ -\mathrm{e}^{\varphi_3}R(z) & I+P^T(z) & -z^m\mathrm{e}^{\varphi_2}R(z) & z^m\mathrm{e}^{-\varphi_1+\varphi_3}P^T(z) \\ -z^{-m}\mathrm{e}^{-\varphi_2+\varphi_3}P(z) & z^{-m}\mathrm{e}^{-\varphi_2}Q(z) & I-P(z) & \mathrm{e}^{-\varphi_1-\varphi_2+\varphi_3}Q(z) \\ -\kappa z^{-m}\mathrm{e}^{\varphi_1}R(z) & \kappa z^{-m}\mathrm{e}^{\varphi_1-\varphi_3}P^T(z) & -\kappa \mathrm{e}^{\varphi_1+\varphi_2-\varphi_3}R(z) & I+\kappa P^T(z) \end{pmatrix} \tag{2.12}$$

where $\kappa = \mathrm{e}^{\alpha}$ and the z dependence of the dual fields is omitted. The integral operators $P(z)$, $Q(z)$, $R(z)$ and $P^T(z)$ are given below:

$$P(z) = \mathrm{i}(q-q^{-1})\frac{qz-1}{z-q}\exp\left(\mathrm{i}q^{-1}\frac{qz-1}{z-q}s - \mathrm{i}q\frac{qz-1}{z-q}t\right) \tag{2.13}$$

$$Q(z) = \mathrm{i}(q-q^{-1})\frac{qz-1}{z-q}\exp\left(\mathrm{i}q^{-1}\frac{qz-1}{z-q}(s+t)\right) \tag{2.14}$$

$$R(z) = \mathrm{i}(q-q^{-1})\frac{qz-1}{z-q}\exp\left(-\mathrm{i}q\frac{qz-1}{z-q}(s+t)\right) \tag{2.15}$$

$$P^T(z) = \mathrm{i}(q-q^{-1})\frac{qz-1}{z-q}\exp\left(-\mathrm{i}q\frac{qz-1}{z-q}s + \mathrm{i}q^{-1}\frac{qz-1}{z-q}t\right). \tag{2.16}$$

They satisfy the following relations:

$$P(z)P(z) = P(z) \qquad P(z)Q(z) = Q(z) \tag{2.17}$$

$$Q(z)R(z) = P(z) \qquad Q(z)P^T(z) = Q(z) \tag{2.18}$$

$$R(z)P(z) = R(z) \qquad R(z)Q(z) = P^T(z) \tag{2.19}$$

$$P^T(z)R(z) = R(z) \qquad P^T(z)P^T(z) = P^T(z). \tag{2.20}$$

Here these products implicitly contain the integrals for extra variables s, t as for (2.10).

Finding the 4×4 matrix integral operator that satisfies (a)–(c) is referred to as the operator-valued Riemann–Hilbert problem. By using its solution $\chi^{(m)}(z)$ the Fredholm determinant $\mathrm{Det}(1-\frac{1}{2\pi}V^{(m)})$ is computed via the following lemma [6]:

Lemma 2. *The Fredholm determinant* $\mathrm{Det}(1-\frac{1}{2\pi}V^{(m)})$ *obeys*

$$\frac{\mathrm{Det}(1-\frac{1}{2\pi}V^{(m+1)})}{\mathrm{Det}(1-\frac{1}{2\pi}V^{(m)})} = \mathrm{Det}(\gamma_2[\chi^{(m)}(0)]^{-1}\gamma_2) \tag{2.21}$$

where $\gamma_2 = \frac{1}{2}(1-\sigma^z\otimes 1)$.

Proof. By lemma 5 and (4.7) of [6] we have

$$\frac{\mathrm{Det}(1-\frac{1}{2\pi}V^{(m+1)})}{\mathrm{Det}(1-\frac{1}{2\pi}V^{(m)})} = \exp\mathrm{Tr}\log(\gamma_1+\gamma_2[\chi^{(m)}(0)]^{-1}\gamma_2) \tag{2.22}$$

where $\gamma_1 = \frac{1}{2}(1 + \sigma^z \otimes 1)$. Due to the orthogonality $\gamma_1\gamma_2 = 0$ and $\gamma_1^2 = \gamma_1$, $\gamma_2^2 = \gamma_2$ it follows that

$$\begin{aligned}\exp \operatorname{Tr} \log(\gamma_1 + \gamma_2[\chi^{(m)}(0)]^{-1}\gamma_2) &= \exp \operatorname{Tr}(\log \gamma_1 + \log(\gamma_2[\chi^{(m)}(0)]^{-1}\gamma_2)) \\ &= \exp \operatorname{Tr} \log(\gamma_2[\chi^{(m)}(0)]^{-1}\gamma_2).\end{aligned} \tag{2.23}$$

Using the relation $\log \operatorname{Det} = \operatorname{Tr} \log$ we obtain the lemma. □

On the basis of this lemma we derive the large-m asymptotic representation of a two-point correlation function $\langle \sigma_m^z \sigma_1^z \rangle$ from the solution $\chi^{(m)}(z)$ of the operator-valued Riemann–Hilbert problem (a)–(c).

3. Asymptotic form of the two-point correlation function

In this section we compute the large-m asymptotic behaviour of the solution $\chi^{(m)}(z)$ of the operator-valued Riemann–Hilbert problem (a)–(c). By virtue of the relations (2.17)–(2.20) the conjugation matrix $L^{(m)}(z)$ can be decomposed into the product of triangular matrix integral operators as follows:

$$L^{(m)}(z) = M^{(m)}(z)N^{(m)}(z) \tag{3.1}$$

where

$$M^{(m)}(z) = \begin{pmatrix} I & \frac{\kappa}{1+\kappa}\mathrm{e}^{-\varphi_3}Q(z) & -z^m\mathrm{e}^{\varphi_2-\varphi_3}P(z) & \frac{\kappa}{1+\kappa}z^m\mathrm{e}^{-\varphi_1}Q(z) \\ 0 & I & -\frac{1}{\kappa}z^m\mathrm{e}^{\varphi_2}R(z) & \frac{1}{1+\kappa}z^m\mathrm{e}^{-\varphi_1+\varphi_3}P^T(z) \\ 0 & 0 & I & \frac{1}{1+\kappa}\mathrm{e}^{-\varphi_1-\varphi_2+\varphi_3}Q(z) \\ 0 & 0 & 0 & I \end{pmatrix} \tag{3.2}$$

and

$$N^{(m)}(z) = \begin{pmatrix} I - \frac{\kappa}{1+\kappa}P(z) & 0 & 0 & 0 \\ -\frac{1}{\kappa}\mathrm{e}^{\varphi_3}R(z) & I + \frac{1}{\kappa}P^T(z) & 0 & 0 \\ -\frac{1}{1+\kappa}z^{-m}\mathrm{e}^{-\varphi_2+\varphi_3}P(z) & \frac{1}{1+\kappa}z^{-m}\mathrm{e}^{-\varphi_2}Q(z) & I - \frac{1}{1+\kappa}P(z) & 0 \\ -\kappa z^{-m}\mathrm{e}^{\varphi_1}R(z) & \kappa z^{-m}\mathrm{e}^{\varphi_1-\varphi_3}P^T(z) & -\kappa\mathrm{e}^{\varphi_1+\varphi_2-\varphi_3}R(z) & I + \kappa P^T(z) \end{pmatrix}. \tag{3.3}$$

Let Γ_1 (Γ_2) be a contour that runs from $\mathrm{e}^{-\mathrm{i}\psi}$ to $\mathrm{e}^{\mathrm{i}\psi}$ inside (outside) the unit circle. Consider a matrix integral operator $\tilde{\chi}^{(m)}(z)$ defined below, as for the case of the lattice sine–Gordon model [9].

(i) $\tilde{\chi}^{(m)}(z) = \chi^{(m)}(z)$ outside the region enclosed by the contours Γ_1 and Γ_2.
(ii) $\tilde{\chi}^{(m)}(z) = \chi^{(m)}(z)M^{(m)}(z)$ in the region enclosed by Γ and Γ_1.
(iii) $\tilde{\chi}^{(m)}(z) = \chi^{(m)}(z)(N^{(m)}(z))^{-1}$ in the region enclosed by Γ and Γ_2.

By definition $\tilde{\chi}^{(m)}(z)$ is analytic for $z \in \mathbb{C}\backslash(\Gamma_1 \cup \Gamma_2)$ and obeys

$$\tilde{\chi}_-^{(m)}(z) = \tilde{\chi}_+^{(m)}(z)M^{(m)}(z) \qquad (z \in \Gamma_1) \tag{3.4}$$

$$\tilde{\chi}_-^{(m)}(z) = \tilde{\chi}_+^{(m)}(z)N^{(m)}(z) \qquad (z \in \Gamma_2). \tag{3.5}$$

We are interested in the asymptotic behaviour of the solution $\chi^{(m)}(z)$ for sufficiently large m. Notice that in the limit $m \to \infty$ the conjugation matrices $M^{(m)}(z)$ and $N^{(m)}(z)$ become block diagonal in the vicinities of contours Γ_1 and Γ_2, respectively. Take this limit and consider the original conjugation matrix for $\chi^{(m)}(z)$ again. Then we have

$$\chi^{(\infty)}(z) = \begin{pmatrix} \Phi_1(z) & 0 \\ 0 & \Phi_2(z) \end{pmatrix}. \tag{3.6}$$

The 2×2 matrix integral operators $\Phi_1(z)$, $\Phi_2(z)$ satisfy

$$\Phi_a^-(z) = \Phi_a^+(z) G_a(z) \qquad (z \in \Gamma, a = 1, 2). \tag{3.7}$$

The corresponding conjugation matrices are expressed by

$$G_1(z) = \begin{pmatrix} I - P(z) & \mathrm{e}^{-\varphi_3} Q(z) \\ -\frac{1}{\kappa}\mathrm{e}^{\varphi_3} R(z) & I + \frac{1}{\kappa} P^T(z) \end{pmatrix} \tag{3.8}$$

$$G_2(z) = \begin{pmatrix} I - P(z) & \mathrm{e}^{-\varphi_1-\varphi_2+\varphi_3} Q(z) \\ -\kappa \mathrm{e}^{\varphi_1+\varphi_2-\varphi_3} R(z) & I + \kappa P^T(z) \end{pmatrix}. \tag{3.9}$$

The calculation of a two-point correlation function $\langle \sigma_m^z \sigma_1^z \rangle$ requires the Fredholm determinant of the solution $\chi^{(m)}(z)$. Taking the Fredholm determinants on both sides of (3.7) we obtain the scalar Riemann–Hilbert problems:

$$\mathrm{Det}\Phi_a^-(z) = \mathrm{Det}\Phi_a^+(z)\mathrm{Det}G_a(z) \qquad (z \in \Gamma, a = 1, 2). \tag{3.10}$$

In the same way as [9] the Fredholm determinants of $G_1(z)$, $G_2(z)$ are computed as

$$\mathrm{Det}G_1(z) = \kappa^{-1} \qquad \mathrm{Det}G_2(z) = \kappa. \tag{3.11}$$

According to the condition at infinity, $\mathrm{Det}\Phi_1(\infty) = \mathrm{Det}\Phi_2(\infty) = 1$, the solutions of scalar problems (3.10) are determined as follows:

$$\mathrm{Det}\Phi_1(z) = \left(\frac{z - \mathrm{e}^{-\mathrm{i}\psi}}{z - \mathrm{e}^{\mathrm{i}\psi}}\right)^{\mathrm{i}\alpha/2\pi} \qquad \mathrm{Det}\Phi_2(z) = \left(\frac{z - \mathrm{e}^{\mathrm{i}\psi}}{z - \mathrm{e}^{-\mathrm{i}\psi}}\right)^{\mathrm{i}\alpha/2\pi}. \tag{3.12}$$

They are analytic for $z \in \mathbb{C}\backslash\Gamma$.

We evaluate the m dependence of the solution $\chi^{(m)}(z)$. Based on the decomposition (3.1) the conjugation matrix for sufficiently large m is expressed as follows:

$$L^{(m)}(z) = \begin{pmatrix} G_1(z) & z^m G_{12}(z) \\ z^{-m} G_{21}(z) & G_2(z) \end{pmatrix} \tag{3.13}$$

where $G_{12}(z)$, $G_{21}(z)$ are 2×2 matrix integral operators independent of m. Let the solution $\chi^{(m)}(z)$ be denoted by

$$\chi^{(m)}(z) = \begin{pmatrix} X_1^{(m)}(z) & X_{12}^{(m)}(z) \\ X_{21}^{(m)}(z) & X_2^{(m)}(z) \end{pmatrix}. \tag{3.14}$$

Here $X_{12}^{(m)}(z)$, $X_{21}^{(m)}(z)$ decay sufficiently rapidly as $m \to \infty$. By the relation (2.11) of the original Riemann–Hilbert problem, the 2×2 matrix integral operators $X_1^{(m)}(z)$, $X_2^{(m)}(z)$ obey

$$\begin{aligned} [X_1^{(m)}(z)]_- &= [X_1^{(m)}(z)]_+ G_1(z) + z^{-m}[X_{12}^{(m)}(z)]_+ G_{21}(z) \\ [X_2^{(m)}(z)]_- &= [X_2^{(m)}(z)]_+ G_2(z) + z^{m}[X_{21}^{(m)}(z)]_+ G_{12}(z) \end{aligned} \qquad (z \in \Gamma). \tag{3.15}$$

In terms of the relation (3.7) the conjugation matrix $G_a(z)$ is given by $[\Phi_a^+(z)]^{-1}\Phi_a^-(z)$ $(a = 1, 2)$. Applying these we have

$$[X_1^{(m)}(z)]_-[\Phi_1^-(z)]^{-1} = [X_1^{(m)}(z)]_+[\Phi_1^+(z)]^{-1} + z^{-m}[X_{12}^{(m)}(z)]_+ G_{21}(z)[\Phi_1^-(z)]^{-1} \tag{3.16}$$

$$[X_2^{(m)}(z)]_-[\Phi_2^-(z)]^{-1} = [X_2^{(m)}(z)]_+[\Phi_2^+(z)]^{-1} + z^{m}[X_{21}^{(m)}(z)]_+ G_{12}(z)[\Phi_2^-(z)]^{-1}. \tag{3.17}$$

Here the variable z lies on the contour Γ. We remark that the inverse of $\Phi_a(z)$ exists because $\mathrm{Det}\Phi_a(z)$ is non-zero for $z \in \Gamma$ $(a = 1, 2)$ (see (3.12)). By means of the Plemelj formulae (see [10]) and the condition $X_1^{(m)}(\infty) = X_2^{(m)}(\infty) = I$, the 2×2 matrix integral operators $X_1^{(m)}$, $X_2^{(m)}(z)$ are expressed below:

$$X_1^{(m)}(z) = \left(I - \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{w^{-m}\,\mathrm{d}w}{w - z}[X_{12}^{(m)}(w)]_+ G_{21}(w)[\Phi_1^-(w)]^{-1}\right)\Phi_1(z) \tag{3.18}$$

$$X_2^{(m)}(z) = \left(I - \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{w^{m}\,\mathrm{d}w}{w - z}[X_{21}^{(m)}(w)]_+ G_{12}(w)[\Phi_2^-(w)]^{-1}\right)\Phi_2(z). \tag{3.19}$$

A two-point correlation function $\langle\sigma_m^z\sigma_1^z\rangle$ is computed by using only $[\chi^{(m)}(0)]^{-1}$. In the case $z = 0$ the right-hand sides must yield some terms whose order is less than $1/m$, because the contour Γ is a part of the unit circle and $X_{12}^{(m)}(z)$, $X_{21}^{(m)}(z)$ decay sufficiently rapidly as $m \to \infty$. Hence we obtain the large-m asymptotic behaviour of the solution $\chi^{(m)}(0)$ as follows:

$$\chi^{(m)}(0) = \begin{pmatrix} \Phi_1(0)(I + \mathrm{O}(m^{-1})) & 0 \\ 0 & \Phi_2(0)(I + \mathrm{O}(m^{-1})) \end{pmatrix} \qquad (m \gg 1). \tag{3.20}$$

We are in a position to compute a two-point correlation function $\langle\sigma_m^z\sigma_1^z\rangle$. Applying the expressions (3.12) and (3.20) to lemma 2 we have

$$\frac{\mathrm{Det}(1 - \frac{1}{2\pi}V^{(m+1)})}{\mathrm{Det}(1 - \frac{1}{2\pi}V^{(m)})} = \exp\left(\frac{\psi}{\pi}\alpha\right)(1 + \mathrm{O}(m^{-1})) \qquad (m \gg 1). \tag{3.21}$$

Since the leading term $\exp(\psi\alpha/\pi)$ does not contain any dual field, the vacuum expectation value of the left-hand side can be written as

$$\frac{Q^{(m+1)}(\alpha)}{Q^{(m)}(\alpha)} = \exp\left(\frac{\psi}{\pi}\alpha\right)(1 + \mathrm{O}(m^{-1})) \qquad (m \gg 1). \tag{3.22}$$

From this recursion relation we derive the large-m asymptotic form of the generating functional as follows:

$$Q^{(m)}(\alpha) = f(m, \alpha)\exp\left(\frac{\psi}{\pi}\alpha m\right) \qquad (m \gg 1). \tag{3.23}$$

Here $f(m, \alpha)$ satisfies $f(m+1, \alpha)/f(m, \alpha) = 1 + \mathrm{O}(m^{-1})$. Substituting this asymptotic form into the relation (2.1) we arrive at the following relation for two-point correlation functions:

$$\frac{\langle\sigma_{m+1}^z\sigma_1^z\rangle}{\langle\sigma_m^z\sigma_1^z\rangle} = 1 + \mathrm{O}(m^{-1}) \qquad (m \gg 1). \tag{3.24}$$

Thus the asymptotic form of a two-point correlation function of the *XXZ* spin chain is shown to be given by power-functions of the distance between spin operators. This result is nothing but the theorem in section 1.

4. Concluding remarks

We have analysed the operator-valued Riemann–Hilbert problem associated with the generating functional of correlation functions and have obtained the large-m asymptotic form of a two-point correlation function $\langle\sigma_m^z\sigma_1^z\rangle$ of the *XXZ* spin chain. This correlation function is exactly expressed in terms of power-functions. Our approach based on Riemann–Hilbert problems is useful for the analysis of long-distance correlations. In a forthcoming publication we will determine the exponent of power, calculating the integrals (3.18) and (3.19). The exponent is expected to depend on the anisotropy parameter η and the external magnetic field h.

As stated in section 1, a special correlation function FSFP is obtained from the generating functional by taking the limit $\alpha \to -\infty$. However, in this limit, our asymptotic expression (3.23) loses its meaning. A similar situation arises from the asymptotic analysis of FSFP of the *XXX* spin chain ($\Delta = 1$) [11]. From the viewpoint of the Riemann–Hilbert problem the evaluation of FSFP is more difficult than that of the generating functional. It is an open problem to compute the asymptotic form of FSFP for the *XXX* and *XXZ* spin chain. It is worth mentioning that at the free fermion point $\Delta = 0$ the corresponding FSFP has been shown to decay as a Gaussian [12].

In this paper we have deformed an operator-valued Riemann–Hilbert problem for 4×4 matrix integral operators to a scalar problem, which has been studied in detail [10]. In general, it is hard to solve an operator-valued Riemann–Hilbert problem: the existence and uniqueness of its solution have not yet been proved. These are fundamental and unsolved questions. The investigation of the operator-valued Riemann–Hilbert problem is important since it contains many applications to mathematics and physics.

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